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Some measures of robustness for unbiased estimators in one-parameter natural exponential families with quadratic variance function

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Abstract

In this paper two measures to highlight the possible effect of an observation on the UMVU estimate are proposed. Our study is based in expansions in terms of orthogonal polynomials for the UMVUE when sampling from a NEF-QVF. We obtain the conditional bias and the asymptotic mean sensitivity curve (AMSC) for the UMVUE. We observe that these measures depend on parametric function under consideration at the true and unknown value of the parameter. We study in detail their properties and relationships as well as to the Hampel's influence function. In fact, we note that the AMSC also verifies for the UMVUE in the NEF-QVF some of most relevant properties of influence function. Also a case-deletion influence diagnostic and some simulations are included to illustrate our results.

Keywords: AMSC; Asymptotic variance; Conditional bias; Influence function; Orthogonal polynomials; NEF-QVF; UMVUE

1. Introduction

The term influence is used in statistics in two different contexts: the influence function (IF) and the influence analysis. Both aspects are closely related. The *influence function* was introduced by Hampel in (1968, 1974) (see [1]). This is a key tool to assess the robustness of an estimator. On the other hand, in *influence analysis* the aim is to provide measures (*influence diagnostics*) to assess the sensitivity of the conclusions of a statistical analysis with respect to minor perturbations in the model or data. The interest of our study is based in the following fact: papers handling influence analysis topics can be found in nearly all statistical techniques and models, however, there are scarcely references handling influence analysis in parametric models. As illustrations we can cite: Kim [2] studied local influence in multivariate normal data, Poon and Tang [3] developed influence measures for parametric models of lifetime data, and Nyangoma, Fung and Jansen [4] in the multinomial model.

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In this paper we get results for the uniformly minimum variance unbiased estimator (UMVUE) in certain parametric models: Natural Exponential Family with Quadratic Variance Function (NEF-QVF). Our study is based in expansions in terms of orthogonal polynomials for the UMVUE. This technique has been used in statistical inference mainly connected with exponential families: Abbey and David [5], Morris [6,7], López-Blázquez [8,9]. Also it can be used in more general settings, see for instance Voinov and Nikulin [10], in nonregular distributions: Barranco-Chamorro et al. [11], and in bayesian statistics: Pommeret [12]. So in this sense, it can be said that we propose a new use of this technique to get results on influence analysis for the UMVUE in NEFs-QVF. The proposed measures are the conditional bias and the asymptotic mean sensitivity curve. They are connected with both aspects of term influence previously mentioned (IF and influence analysis).

The *conditional bias* was introduced by Muñoz-Pichardo et al. [13]. This measure assesses the effect on an estimator T_n of a fixed sample value x through the difference between the expected value of T_n and the conditional expectation of this estimator given this observation (for instance $X_1 = x$), i.e.

$$E_\theta[T_n|X_1 = x] - E_\theta[T_n]. \tag{1}$$

(1) has been applied in a variety of fields in statistics as an influence diagnostic. As examples we can cite the papers by Muñoz-Pichardo et al. [14] in multivariate linear general models, Jiménez-Gamero et al. [15] in regression with complex designs, and Moreno-Rebollo et al. [16] in survey sampling.

The IF is a powerful tool in robustness, however it has the disadvantage of being built upon statistical functionals. Taking into account this disadvantage Schlittgen and Schwabe [17] introduced the *asymptotic mean sensitivity curve* (AMSC) as an alternative to the IF based on the asymptotic version of the Tukey's sensitivity curve using only expectations and limits, that is

$$\mathrm{AMSC}(x; T_n, \theta) = \lim_{n\to\infty} n\big\{E_\theta[T_n|X_1 = x] - E_\theta[T_n]\big\}, \tag{2}$$

provided this limit exists and T_n is a consistent sequence of estimators of the parameter θ. The AMSC has a clear meaning, is easy to introduce at an intermediate level, and often agrees with the Hampel's IF. Moreover the conditional bias and the AMSC are obviously related.

From now on we focus on the conditional bias and the AMSC for the UMVUE in the NEF-QVF. Common one-parameter distributions belong to the NEF-QVF: normal, gamma, binomial, Poisson, ... For these distributions UMVUEs are frequently available and can be used in a great number of practical applications (see for instance Voinov and Nikulin [10]). UMVUEs in the NEF-QVF rely on the sample mean. The sample mean is well known to be highly nonrobust since its IF is unbounded. So the issue of UMVUEs' robustness in these distributions raises in a natural way. Our aim is to highlight the possible effect of an observation on the UMVU estimate when sampling from a NEF-QVF.

The *outline* is the following one: in Section 2 we give some preliminaries about the NEF-QVF and properties of a system of orthogonal polynomials associated with them. In Section 3 the conditional bias and the AMSC for the UMVUE of a given parametric function are obtained. Some comments about the meaning and relationships between the proposed measures are given in Section 3.4. In Section 3.5 we see that, under asymptotic normality, the AMSC is related to the asymptotic variance of the UMVUE. In Section 4 we illustrate how an unbiased estimator for the conditional bias can be considered as a case-deletion influence diagnostic. Some simulations are carried out in which we show that the proposed diagnostic identify different influential observations depending on parametric function under consideration and the true value of the parameter. Some final conclusions are given in Section 5.

2. NEF-QVF

For a random variable (rv) X belonging to the one-parameter NEF its probability density function (pdf) with respect to a sigma-finite measure ν on the Borel subsets of $\mathbb{R}$ can be written as

$$f(x;\theta) = \exp\big\{x\theta - \Psi(\theta)\big\}, \quad \theta \in \Theta \subseteq \mathbb{R}. \tag{3}$$

The natural parameter space Θ is the largest open set for which $\int \exp(x\theta)\, d\nu(x)$ is finite. It will be assumed that Θ is nonempty. The mean and the variance of X are $\mu = E_\theta[X] = \Psi'(\theta)$ and $Var_\theta[X] = \Psi''(\theta)$. Since $\Psi''(\theta) > 0$, it is possible to reparameterize the pdf given in (3) in terms of μ

$$f(x;\mu) = \exp\big\{x\theta(\mu) - \Psi\big(\theta(\mu)\big)\big\}, \quad \mu \in \Omega \subseteq \mathbb{R}, \tag{4}$$

where $\Omega = \Psi'(\Theta)$ is the mean space, and the variance can be seen as a function of μ, $Var_\theta[X] = V(\mu) = \Psi''((\Psi')^{-1}(\mu))$. This is the variance function.

Let $X_1, \dots, X_n$, with $n \geqslant 1$, be a simple random sample (srs) from (4) (i.e. independent and identically distributed rv's). Then $S_n = \sum_{i=1}^n X_i$ is a sufficient and complete statistic for μ whose pdf with respect to the n-fold convolution measure $\nu_n = \nu \times \cdots \times \nu$ is

$$f_{n,\mu}(s) = \exp\{s\theta(\mu) - n\Psi(\theta(\mu))\}, \quad \mu \in \Omega. \tag{5}$$

Let us consider the space of Borel-measurable square integrable functions of S_n,

$$L^2_{n,\mu} = \left\{T_n\colon \int T_n^2(s) f_{n,\mu}(s)\nu_n(ds) < \infty\right\}.$$

For each $\mu \in \Omega$, $L^2_{n,\mu}$ is a Hilbert space with the inner product $\langle T_1, T_2\rangle_{n,\mu} = E_\mu[T_1(S_n)T_2(S_n)]$, and norm induced $\|T_1\|^2_{n,\mu} = E_\mu[T_1^2(S_n)]$, $T_1, T_2 \in L^2_{n,\mu}$. As usual in the theory of L^2-spaces, two functions $T_1, T_2 \in L^2_{n,\mu}$ will be considered as equivalent if $T_1(S_n) = T_2(S_n)$ ν_n-a.s. (i.e. $\nu_n\{T_1 \neq T_2\} = 0$).

We concentrate on NEFs whose variance function is, at most, a quadratic function of the mean μ, $V(\mu) = v_0 + v_1\mu + v_2\mu^2$ with $v_i \in \mathbb{R}$. They are denoted as NEF-QVF(μ, $V(\mu)$). NEFs-QVF are six one-parameter families listed in Table 1 (and linear functions of them). Details can be seen in Morris [6,7]. Given a srs from a NEF-QVF(μ, $V(\mu)$), S_n follows a NEF-QVF($n\mu$, $nV(\mu)$). The importance of having a quadratic variance function is that in this case:

(i) An orthogonal polynomial system (OPS) on $L^2_{n,\mu}$ is given by

$$p_{j,n}(s;\mu) = V^j(\mu)\left(\frac{d^j}{d\mu^j} f_n(s;\mu)\right)\frac{1}{f_n(s;\mu)}, \quad j \geqslant 0. \tag{6}$$

In particular: $p_{0,n}(s;\mu) = 1$, $p_{1,n}(s;\mu) = (s - n\mu)$, and $p_{2,n}(s;\mu) = (s - n\mu)^2 - V'(\mu)(s - n\mu) - nV(\mu)$.

(ii) Every $T_n \in L^2_{n,\mu}$ admits an expansion in terms of the OPS $\{p_{j,n}\}_{j\geqslant 0}$

$$T_n(s) = \sum_{j=0}^{\infty} a_{j,n}(\mu) p_{j,n}(s;\mu) \quad \text{with } a_{j,n}(\mu) = \frac{\langle T_n, p_{j,n}\rangle_{n,\mu}}{\|p_{j,n}\|^2_{n,\mu}}, \ \mu \in \Omega. \tag{7}$$

Moreover given T_n belongs to $L^2_{n,\mu}$ if and only if the coefficients $a_{j,n}(\mu)$ previously defined verify

$$\sum_{j=0}^{\infty} a^2_{j,n}(\mu)\|p_{j,n}\|^2_{n,\mu} < \infty,$$

and in this case the series $\sum_{j=0}^{\infty} a_{j,n}(\mu)p_{j,n}(\cdot;\mu)$ converges in $L^2_{n,\mu}$-sense to T_n (Abbey and David [5]).

We list below some results that provide us proper tools to get influence measures for the UMVUE of a given parametric function in the NEF-QVF.

Lemma 1. *The polynomials defined in* (6) *verify the following properties*:

(i) *$p_{k,n}$ is a polynomial in $(s - n\mu)$ of degree k with leading term $(s - n\mu)^k$.*

(ii) *Orthogonality relation*

$$E_\mu[p_{k,n}(S_n;\mu)p_{j,n}(S_n;\mu)] = \delta_{kj}\, j!\beta_{j,n}V^j(\mu), \tag{8}$$

δ_{kj} is the Kronecker delta, $\beta_{j,n} = \prod_{i=0}^{j-1}(n + iv_2)$ for $j \geqslant 1$, and $\beta_{0,n} = 1$.

(iii) *For any positive integers $1 \leqslant m \leqslant n$*

$$E_\mu[p_{k,n}(S_n;\mu) \mid S_m] = p_{k,m}(S_m;\mu) \quad (a.s.), \quad k \geqslant 0. \tag{9}$$

Proof. (i) and (ii) can be easily deduced from Morris [6, Section 8]. (iii) can be seen in López-Blázquez and Salamanca-Miño [9]. □

Remark 2. From (8) $E_\mu[p_{k,n}(S_n;\mu)] = \delta_{k0}$, i.e. all the polynomials, but $p_{0,n}$, have mean zero, and their norms are $\|p_{k,n}\|^2_{n,\mu} = E_\mu[p^2_{k,n}] = k!\beta_{k,n}V^k(\mu)$. In particular $\|p_{1,n}\|^2_{n,\mu} = nV(\mu)$ and $\|p_{2,n}\|^2_{n,\mu} = 2n(n+v_2)V^2(\mu)$.

3. Effect of an observation on the UMVUE in the NEF-QVF

We propose two measures of robustness for T_n, the UMVUE of a given parametric function when sampling from a NEF-QVF: (i) the *conditional bias* of T_n caused by the presence of an observation x in the sample and (ii) the AMSC proposed in [17]. First, we recall some properties of the UMVUE. We follow the techniques proposed by Abbey and David [5] and Morris [6,7] for constructing unbiased estimators in the exponential and NEF-QVF distributions.

3.1. UMVUE and NEF-QVF

In the NEF-QVF S_n is a complete sufficient statistic for μ, so if there exists for a sample size n the UMVUE, T_n, of a parametric function $h(\mu)$, then from the Lehmann–Scheffé Theorem $T_n = T_n(S_n)$. Also recall that if $T_n = \text{UMVUE}_n(h(\mu))$, then $T_n \in L^2_{n,\mu}$ for all $\mu \in \Omega$. Therefore T_n admits an expansion as the one that was given in (7). Specifically, the expansion of T_n in terms of the OPS introduced in (6) is

$$T_n(S_n) = \sum_{j=0}^{\infty} \frac{h^{(j)}(\mu)}{j!\beta_{j,n}} p_{j,n}(S_n;\mu) \quad \text{(a.s.)}, \tag{10}$$

and the variance of T_n is

$$Var_\mu\big[T_n(S_n)\big] = \sum_{j=1}^{\infty} a^2_{j,n}(\mu)\|p_{j,n}\|^2_{n,\mu} = \sum_{j=1}^{\infty} \frac{\{h^{(j)}(\mu)\}^2}{j!\beta_{j,n}} V^j(\mu) < \infty, \quad \forall\mu \in \Omega. \tag{11}$$

Let us denote by $\mathcal{U}_n$ the set of UMVU-estimable functions for a given sample size n. In the NEF-QVF, $\mathcal{U}_n$ is a subset of the set of analytic functions in Ω satisfying (11) (Abbey and David [5]).

3.2. Conditional bias for the UMVUE to an observation

The conditional bias of $T_n = \text{UMVUE}_n(h(\mu))$ will be denoted by $b(T_n, h(\mu)|x)$. Without loss of generality, it can be supposed that the fixed observation is $X_1 = x$. From (1) and the unbiasedness of T_n (i.e. $E_\mu[T_n(S_n)] = h(\mu)$)

$$b\big(T_n, h(\mu)\big|x\big) = E_\mu[T_n|X_1 = x] - h(\mu). \tag{12}$$

Theorem 3. *Let $h \in \mathcal{U}_n$ and $T_n(S_n) = \text{UMVUE}_n(h(\mu))$. Then for almost every $x \in \text{support}(X_1)$ with respect to $f_{1,\mu}$, we have*

$$b\big(T_n, h(\mu)\big|x\big) = \sum_{j=1}^{\infty} \frac{h^{(j)}(\mu)}{j!\beta_{j,n}} p_{j,1}(x;\mu), \quad \forall\mu \in \Omega. \tag{13}$$

Proof. From the expression of T_n given in (10) and applying Lemma 1(iii)

$$E_\mu\big[T_n(S_n)\big|X_1\big] = \sum_{j=0}^{\infty} \frac{h^{(j)}(\mu)}{j!\beta_{j,n}} E_\mu\big[p_{j,n}(S_n;\mu)\big|X_1\big] = \sum_{j=0}^{\infty} \frac{h^{(j)}(\mu)}{j!\beta_{j,n}} p_{j,1}(X_1;\mu) \quad \text{(a.s.)}.$$

From properties of conditional expectation and since $T_n \in L^2_{n,\mu}$, it follows that $g_{n,h,\mu}(X_1) = E_\mu[T_n(S_n)|X_1] \in L^2_{1,\mu}$. Therefore for almost every (a.e.) $x \in \text{support}(X_1)$ w.r.t. $f_{1,\mu}$, we can write

$$E_\mu\big[T_n(S_n)\big|X_1 = x\big] = \sum_{j=0}^{\infty} \frac{h^{(j)}(\mu)}{j!\beta_{j,n}} p_{j,1}(x;\mu), \quad \forall\mu \in \Omega. \tag{14}$$

Taking into account that $\beta_{0,n} = 1$, $p_{0,1} = 1$, and (12), we get (13). $\square$

Table 1
Conditional bias and AMSC for the UMVUE of $V(\mu)$ in NEF-QVF distributions

NEF-QVF distributions	
Normal $N(\mu,\sigma^2)$, $\sigma^2>0$ known, $\Omega=(-\infty,\infty)$, $V(\mu)=\sigma^2$ (constant variance function)	$b(T_n,V(\mu)\|x)=0$ $\mathrm{AMSC}(x;T_n,V(\mu))=0$
Poisson $Po(\mu)$, $\Omega=(0,\infty)$, $V(\mu)=\mu$ (linear variance function)	$b(T_n,V(\mu)\|x)=\frac{1}{n}(x-\mu)$ $\mathrm{AMSC}(x;T_n,V(\mu))=x-\mu$, $\mu\in\Omega$
Gamma $G(r,\lambda)$, $r>0$ known, $\mu=r\lambda$, $\Omega=(0,\infty)$, $V(\mu)=\mu^2/r$ (quadratic variance function)	$b(T_n,V(\mu)\|x)=\frac{1}{nr+1}\{\frac{(x-\mu)^2}{n}+2\mu(x-\mu)-\frac{\mu^2}{nr}\}$ $\mathrm{AMSC}(x;T_n,V(\mu))=\frac{2\mu}{r}(x-\mu)$, $\mu\in\Omega$
Binomial $B(r,p)$, $r\in Z^+$ known $\mu=rp$, $\Omega=(0,r)$, $V(\mu)=-\mu^2/r+\mu$ (quadratic variance function)	$b(T_n,V(\mu)\|x)$ $=-\frac{1}{nr-1}\{\frac{(x-\mu)^2}{n}+(2\mu-r)(x-\mu)+\frac{\mu(\mu-r)}{nr}\}$ $\mathrm{AMSC}(x;T_n,V(\mu))=(1-\frac{2\mu}{r})(x-\mu)$, $\mu\in\Omega-\{r/2\}$
Negative binomial $NB(r,p)$, $r\in Z^+$ known $\mu=r(1-p)/p$, $\Omega=(0,\infty)$, $V(\mu)=\mu^2/r+\mu$ (quadratic variance function)	$b(T_n,V(\mu)\|x)$ $=\frac{1}{nr+1}\{\frac{(x-\mu)^2}{n}+(r+2\mu)(x-\mu)-\frac{\mu(\mu+r)}{nr}\}$ $\mathrm{AMSC}(x;T_n,V(\mu))=(1+\frac{2\mu}{r})(x-\mu)$, $\mu\in\Omega$
Generalized hyperbolic secant $GHS(r,\lambda)$, $r>0$ known $\mu=r\lambda$, $\Omega=(-\infty,\infty)$, $V(\mu)=\mu^2/r+r$ (quadratic variance function)	$b(T_n,V(\mu)\|x)$ $=\frac{1}{nr+1}\{\frac{(x-\mu)^2}{n}+2\mu(x-\mu)-\frac{\mu^2+r^2}{nr}\}$ $\mathrm{AMSC}(x;T_n,V(\mu))=\frac{2\mu}{r}(x-\mu)$, $\mu\in\Omega-\{0\}$

Example 4. Let $X_1,\dots,X_n$ be a srs from a NEF-QVF(μ,$V(\mu)$).

(a) Let $h(\mu)=\mu$. Then $T_n(S_n)=\bar{X}_n$ and $b(\bar{X}_n,\mu|x)=\frac{1}{n}(x-\mu)$. Note that $b(\bar{X}_n,\mu|x)$ is a linear function on $(x-\mu)$.
(b) Let $h(\mu)=V(\mu)=v_0+v_1\mu+v_2\mu^2$, with $v_i\in\mathbb{R}$. Then $T_n(S_n)=\frac{n}{n+v_2}V(\bar{X}_n)$ and $b(T_n,V(\mu)|x)=\frac{1}{n+v_2}\{\frac{v_2}{n}(x-\mu)^2+V'(\mu)(x-\mu)-\frac{v_2}{n}V(\mu)\}$. In this case we have quadratic functions on $(x-\mu)$ varying with the distribution in the NEF-QVF. The explicit expressions are given in Table 1.

Remark 5. If $v_2<0$ (i.e. binomial $B(r,p)$ distribution), then the SPO given in (6) is finite ($p_{j,n}=0$ for $j>N$, $N=rn$). In this case only polynomials in μ of degree $\leqslant N$ admit an unbiased estimator, and therefore the series for T_n and $b(T_n,h(\mu)|x)$ have only a finite number of terms.

3.3. Asymptotic mean sensitivity curve for the UMVUE

Next the AMSC proposed by Schlittgen and Schwabe [17] as an alternative to the IF is studied for $T_n(S_n)$ the UMVUE of a given $h(\mu)$. Since $T_n(S_n)$ is a consistent sequence of estimators for $h(\mu)$, (2) can be applied. Also note that from (1) and (2): $\mathrm{AMSC}(x;T_n,h(\mu))=\lim_{n\to\infty}nb(T_n,h(\mu)|x)$.

Remark 6. Recall that if $h(\mu)$ is a UMVU-estimable function for a sample size n_0 with UMVUE $T_{n_0}(S_{n_0})$, then $h\in\mathcal{U}_n$ for $n\geqslant n_0$ and its UMVUE is $T_n(S_n)$. So the condition expressed in this subsection as $h\in\mathcal{U}_n$ must be understood as $h\in\mathcal{U}_n$ for $n\geqslant n_0$ (certain n_0).

Let us denote by $\xrightarrow{\mathcal{L}_2}$ convergence in mean of order two.

Theorem 7. *Let $h\in\mathcal{U}_n$ and $T_n(S_n)=\mathrm{UMVUE}_n(h(\mu))$. Then*

$$nb\big(T_n,h(\mu)\big|X\big)\xrightarrow{\mathcal{L}_2}h'(\mu)(X-\mu),\quad \mu\in\Omega. \tag{15}$$

Proof. For $\mu \in \Omega$, from (13) and the orthogonality of $\{p_{j,1}\}$

$$\left\| nb\bigl(T_n, h(\mu)\big|X\bigr) - h'(\mu)(X-\mu)\right\|^2_{1,\mu} = n^2 \sum_{j=2}^{\infty} \frac{\{h^{(j)}(\mu)\}^2}{(j!\beta_{j,n})^2} \|p_{j,1}\|^2_{1,\mu}.$$

$b(T_n, h(\mu)|X) \in L^2_{1,\mu}$, therefore the previous series is convergent. Moreover this is a series of positive terms, so limit and sum can be interchanged. Also recall that $\beta_{j,n} = \prod_{i=0}^{j-1}(n + iv_2)$, therefore for $v_2 = 0$ and $v_2 > 0$ we have that for $j \geqslant 2$ $\lim_{n\to\infty}(n/\beta_{j,n})^2 = 0$ and

$$\lim_{n\to\infty} \left\| nb\bigl(T_n, h(\mu)\big|X\bigr) - h'(\mu)(X-\mu)\right\|^2_{1,\mu} = 0. \tag{16}$$

If $v_2 < 0$ (binomial distribution), then from Remark 5, $b(T_n, h(\mu)|X)$ has only a finite number of terms and we have (16) immediately. In any case from definition of convergence in mean of order two from (16), it follows (15). □

Note that Theorem 7 provides the limit of $b(T_n, h(\mu)|X)$ by using the usual metric in $L^2_{1,\mu}$. In the following we will denote (15) by

$$\mathrm{AMSC}\bigl(x; T_n, h(\mu)\bigr) = h'(\mu)(x-\mu), \tag{17}$$

taking into account that in this case, unlike Schlittgen and Schwabe [17], the term asymptotic refers to convergence in mean of order two.

Similarly for any $k \geqslant 1$ with $k = \min\{j \in Z^+ \colon h^{(j)}(\mu) \neq 0\}$, it can be proved

$$n^k b\bigl(T_n, h(\mu)\big|X\bigr) \xrightarrow{\mathcal{L}_2} \frac{h^{(k)}(\mu)}{k!} p_{k,1}(X;\mu), \quad \mu \in \Omega. \tag{18}$$

Analogously to (2), a *generalized asymptotic mean sensitivity curve of order k* (GAMSC_k) for T_n at x and $h(\mu)$ can be proposed

$$\mathrm{GAMSC}_k\bigl(x; T_n, h(\mu)\bigr) = \frac{h^{(k)}(\mu)}{k!} p_{k,1}(x;\mu) \tag{19}$$

provided k is the order of the first nonzero derivative of h at μ ($k \geqslant 2$).

Remark 8. $k = \min\{j \in Z^+ \colon h^{(j)}(\mu) \neq 0\}$ depends on $\mu \in \Omega$, i.e. $k = k(\mu)$.

Remark 9. Note that the AMSC for the UMVUE of $h(\mu)$ given in (17) is a linear function on $(x-\mu)$ with slope $h'(\mu)$ and $(x-\mu)$ is the Hampel's IF of the sample mean.

On the other hand, from Lemma 1(i), $\mathrm{GAMSC}_k(x; T_n, h(\mu))$ given in (19) is a polynomial in $(x-\mu)$ of degree k with leading coefficient $[h^{(k)}(\mu)/k!]$.

Example 10. Let $T_n(S_n)$ be the UMVUE of $V(\mu)$ given in Example 4. Table 1 summarizes $\mathrm{AMSC}(x; T_n, V(\mu))$ for those distributions in the NEF-QVF and $\mu \in \Omega$ such that $V'(\mu) \neq 0$. Moreover for those $\mu \in \Omega$ such that $V'(\mu) = 0$ a GAMSC of order 2 can be given (see $V(\mu)$ and Ω in Table 1)

- Binomial $B(r, p)$ distribution with $p = 1/2$ (i.e. $\mu = r/2$). Then $\mathrm{GAMSC}_2(x; T_n, V(r/2)) = \frac{1}{4} - \frac{1}{r}(x - \frac{r}{2})^2$, $x \in \{0, 1, \ldots, r\}$.
- Generalized hyperbolic secant *GHS*(r, λ) distribution ($r > 0$ known), with $\lambda = 0$ (i.e. $\mu = 0$). Then

$$\mathrm{GAMSC}_2\bigl(x; T_n, V(0)\bigr) = \frac{x^2}{r} - 1, \quad x \in (-\infty, \infty).$$

3.4. Comments on the previous results given in this section

Fixed $\mu \in \Omega$ and $k = k(\mu) = \min\{j \in Z^+ \colon h^{(j)}(\mu) \neq 0\}$, let us look again at (14) and note that this can be written as

$$E_\mu\bigl[T_n(S_n)\big|X_1 = x\bigr] = h(\mu) + \frac{h^{(k)}(\mu)}{k!\beta_{k,n}} p_{k,1}(x;\mu) + \cdots.$$

If $k=1$, $(h'(\mu)\neq 0)$, then $\mathrm{AMSC}(x;T_n,h(\mu))=h'(\mu)(x-\mu)$. From (2) notice that the AMSC gives the term of order $1/n$ at the conditional expectation of T_n to $X_1=x$. Similarly, since $\beta_{k,n}=O(n^{-k})$, the $\mathrm{GAMSC}_k(x;T_n,h(\mu))=[h^{(k)}(\mu)/k!]p_{k,1}(x;\mu)$ gives the first nonzero term at the bias conditional to x (i.e. the one of the order $1/n^k$).

Roughly speaking, both curves give the main effect that an observation x can produce on the conditional bias of T_n (normalized by n^k), with k the order of the first nonzero derivative of h at $\mu\in\Omega$.

Schlittgen and Schwabe [17] showed that some of the most important properties of the Hampel's IF hold for the AMSC if we consider L- and M-estimators. These properties also hold for the AMSC and the UMVUE in the NEF-QVF, but not in general for the GAMSC_k with $k\geqslant 2$ (Section 3.5). This is the main reason why we use the term *AMSC* for $k=1$ and propose the term GAMSC_k for $k\geqslant 2$.

3.5. Relationship between $E[AMSC^2]$ and the asymptotic variance of the UMVUE

The IF and the asymptotic variance of an estimator are related as it can be seen in Hampel et al. [1]. Specifically, if T is a functional that defines an estimator and under conditions of asymptotic normality, then the asymptotic variance of the estimator T at the distribution F, $\mathrm{ASV}(T,F)$, is equal to the moment of order 2 of the influence function

$$\mathrm{ASV}(T,F)=E_F\big[\mathrm{IF}(X;T,F)^2\big].$$

Schlittgen and Schwabe [17] showed that for L- and M-estimators, an analogous relationship is verified for the AMSC and the variance (properly normalized) of the sequence of estimators T_n under consideration. Next, we prove that this relationship also holds between the moment of order 2 of the AMSC and the asymptotic variance of the UMVUE, T_n, of $h\in\mathcal{U}_n$. First, recall the possibilities for the limit distribution of the UMVUE in the NEF-QVF.

Lemma 11 *(Limit distributions of the UMVUE in the NEF-QVF). Let $h\in\mathcal{U}_n$, $T_n(S_n)=\mathrm{UMVUE}_n(h(\mu))$, $\mu\in\Omega$ and $k=\min\{j\in Z^+:\ h^{(j)}(\mu)\neq 0\}$. Then*

$$n^{k/2}\big\{T_n(S_n)-h(\mu)\big\}\xrightarrow{d}\frac{h^{(k)}(\mu)}{k!}V^{k/2}(\mu)H_k(Z),\quad n\to\infty \tag{20}$$

where $\xrightarrow{d}$ denotes convergence in distribution, $Z\sim N(0,1)$, and $H_k(z)$ is the Hermite polynomial of degree k, $k\geqslant 1$.

Proof. It can be seen in López-Blázquez and Castaño-Martínez [8]. □

Remark 12. For $k\geqslant 1$ the Hermite polynomials can be recursively constructed by the relation $H_k(z)=zH_{k-1}(z)-(k-1)H_{k-2}(z)$ with $H_0=1$ and $H_{-1}=0$. The norm of these polynomials is $\|H_k\|^2=k!$. (See Chihara [18].)

Note that for $k=1$ Lemma 11 gives the asymptotic normality of the UMVUE in the NEF-QVF (Portnoy [19]).

Theorem 13. *Let $h\in\mathcal{U}_n$, $T_n(S_n)=\mathrm{UMVUE}_n(h(\mu))$, $\mu\in\Omega$ and $k=\min\{j\in Z^+:\ h^{(j)}(\mu)\neq 0\}$.*

(i) *If $k=1$ (i.e. $h'(\mu)\neq 0$), then*

$$E\big[\mathrm{AMSC}\big(X;T_n(S_n),h(\mu)\big)^2\big]=\big\{h'(\mu)\big\}^2V(\mu). \tag{21}$$

Moreover, (21) *agrees with the asymptotic variance of T_n.*

(ii) *For $k\geqslant 2$*

$$E\big[\mathrm{GAMSC}_k\big(X;T_n,h(\mu)\big)^2\big]=\frac{\{h^{(k)}(\mu)\}^2}{k!}\beta_{k,1}V^k(\mu), \tag{22}$$

where $\beta_{k,1}=\prod_{i=0}^{k-1}(1+iv_2)$.

Proof. (i) (21) follows from the expression of $\mathrm{AMSC}(x;T_n,h(\mu))$ given in (17). From Lemma 11, this is also the asymptotic variance of T_n for $k=1$.

(ii) From (19) $\mathrm{GAMSC}_k(x; T_n, h(\mu)) = \frac{h^{(k)}(\mu)}{k!} p_{k,1}(x; \mu)$. Therefore

$$E\left[\mathrm{GAMSC}_k\left(X; T_n, h(\mu)\right)^2\right] = \left\{\frac{h^{(k)}(\mu)}{k!}\right\}^2 \|p_{k,1}\|^2_{1,\mu} = \frac{\{h^{(k)}(\mu)\}^2}{k!}\beta_{k,1} V^k(\mu). \qquad \square$$

Remark 14. In general (22) does not agree with the asymptotic variance of T_n, since from Lemma 11 and Remark 12 that is $\{h^{(k)}(\mu)\}^2 V^k(\mu)/k!$. Moreover, notice that (22) depends on the family of distributions through v_2 in $\beta_{k,1}$. For the different distributions in the NEF-QVF v_2 can be obtained from Table 1. For instance for the Normal and Poisson distributions (in both cases $v_2 = 0$), (22) always agrees with the asymptotic variance of the UMVUE.

4. Simulations

In Section 3.2 we saw that fixed x, the conditional bias of an estimator is again a function of the unknown parameter μ. From (1) we propose an unbiased estimator for the conditional bias in the next corollary.

Corollary 15. *Let T_n be the* $\mathrm{UMVUE}_n(h(\mu))$. *Then an unbiased estimator of $b(T_n, h(\mu)|x)$ based on $W_{n-1} = \sum_{i=2}^n X_i$ is $T_n(x + W_{n-1}) - T_{n-1}(W_{n-1})$.*

The unbiased estimator of $b(T_n, h(\mu)|x)$ given in Corollary 15 can be rewritten as a *case-deletion influence diagnostic*

$$\widehat{b}_n\left(h(\mu), x\right) = T_n(S_n) - T_{n-1}(S_n - x). \tag{23}$$

In this section our aim is to illustrate how $\widehat{b}_n(h(\mu), x)$ exhibits the theoretical results about the conditional bias given in Section 3.2. First we give theoretical comments about the parametric function h under consideration and how $\widehat{b}_n(h(\mu), x)$ is as a function of given data x's. Later we summarize pertinent facts about some simulations we have carried out.

Theoretical comments. In some cases given a parametric function h we know how $\widehat{b}_n(h(\mu), x)$ is as a function of x. For instance if h is a polynomial in μ of degree r ($r \geqslant 1$), then from (10) the UMVUE of h, T_n is a polynomial in S_n of degree r. From (23) $\widehat{b}_n(h(\mu), x)$ is also a polynomial in x of degree r.

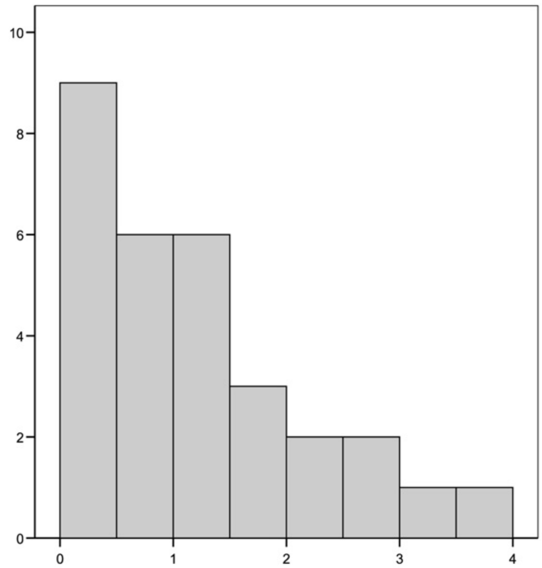
Example 16. Let $h(\mu) = a\mu + b$ with $a, b \in \mathbb{R}$. Then $\widehat{b}_n(h(\mu), x) = a(x - \bar{x}_n)/(n-1)$. Note that $\widehat{b}_n(h(\mu), x)$ is an increasing function of $|x - \bar{x}_n|$.

However, in general, h (as a function of μ) and $\widehat{b}_n(h(\mu), x)$ (as a function of x) can be different types of functions.

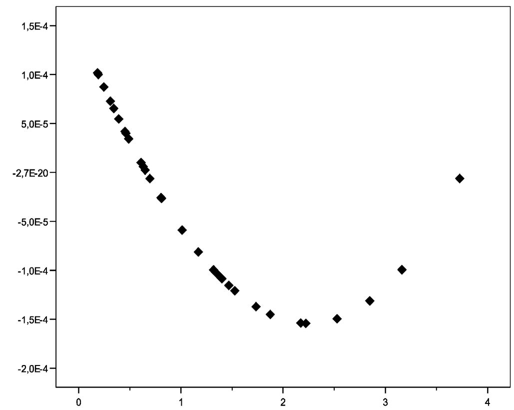
Comments about simulations. We simulated several distributions in the NEF-QVF and considered different kinds of parametric functions $h(\mu)$. For each one we have carried out the scatter-plot showing sample values x versus their conditional bias estimates $\widehat{b}_n(h(\mu), x)$ (i.e. $x - \widehat{b}_n(h(\mu), x)$). In these plots we have only observed a nearly "linear" or "quadratic" behaviour for $\widehat{b}_n(h(\mu), x)$ as a function of given x's (even for different kinds of parametric functions h). Let us denote by μ_0 the value of the parameter considered in each simulation. From our studies we conclude the following facts:

If $h'(\mu_0) \neq 0$, then the corresponding scatter-plot shows a nearly linear relation between x and $\widehat{b}_n(h(\mu), x)$. That is, $\widehat{b}_n(h(\mu), x)$ exhibits the theoretical behaviour of $b(T_n, h(\mu)|x)$ given in Theorems 3 and 7 (the main term in $b(T_n, h(\mu)|x)$ is the linear one).

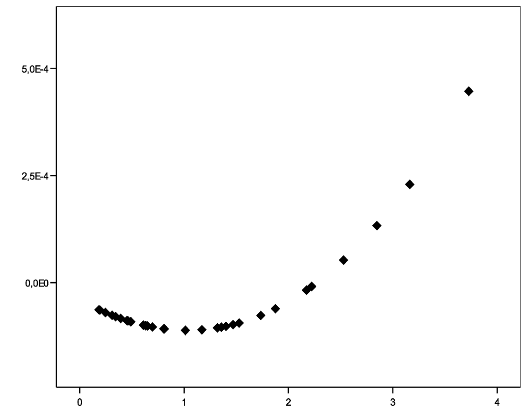
If $h'(\mu_0) = 0$ and $h''(\mu_0) \neq 0$ we have also observed in many cases a linear behaviour for $\widehat{b}_n(h(\mu), x)$ as a function of x. This is apparently contrary to Theorems 3 and 7. Main reason we have found to explain this fact is that quite often the local extremes of $\widehat{b}_n(h(\mu), x)$ are located outside of the range of values in the sample. However, if a local extreme of $\widehat{b}_n(h(\mu), x)$ is near to central values in our sample, then $\widehat{b}_n(h(\mu), x)$ is clearly a parabola as a function of x. Also it is more difficult to see a parabolic behaviour in $\widehat{b}_n(h(\mu), x)$ if the sample size increases. Next example helps us to make clear these considerations.



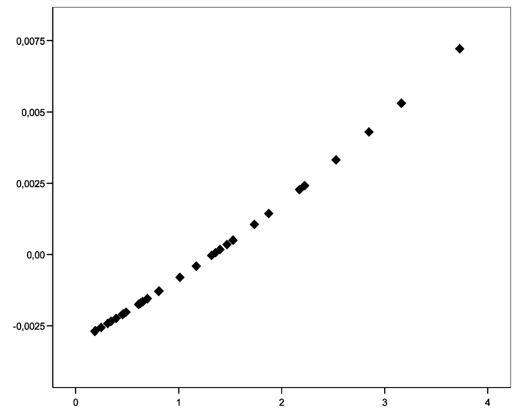
Histogram of simulated data from an exponential distribution



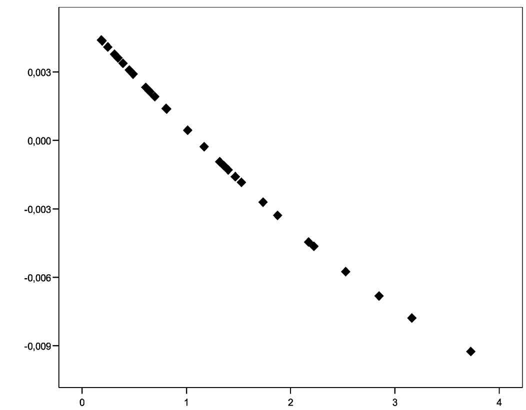
(a)



(b)



(c)



(d)

Fig. 1. Histogram and scatter-plots for the simulation carried out. (a) $x - \widehat{b}_n(h(\mu_0), x)$ for $P_\mu[t_1 < X < t_2]$ with $(t_1, t_2) = (1, 1.5)$; (b) $x - \widehat{b}_n(h(\mu_0), x)$ for $P_\mu[t_1 < X < t_2]$ with $(t_1, t_2) = (1, 1.6)$; (c) $x - \widehat{b}_n(h(\mu_0), x)$ for $P_\mu[t_1 < X < t_2]$ with $(t_1, t_2) = (1, 2.5)$; (d) $x - \widehat{b}_n(h(\mu_0), x)$ for $h_a(\mu) = (\mu - a)^3$.

Example 17. Let $h(\mu)$ a polynomial in μ of degree 2. Then $\widehat{b}_n(h(\mu), x)$ can be expressed as a parabola in x whose vertex is $\bar{x}_n + \frac{V'(\bar{x}_n)}{2} - \frac{h'(\bar{x}_n)}{h''(\bar{x}_n)}(n-1+v_2)$, provided $h''(\bar{x}_n) \neq 0$. This vertex can be located far away from sample values. Especially if $h'(\bar{x}_n) \neq 0$ and n is large enough. Moreover the curvature of $\widehat{b}_n(h(\mu), x)$ at its vertex is given by $-\frac{h''(\bar{x}_n)}{(n-1)(n-1+v_2)}$ and this is a decreasing function of the sample size n.

As a final illustration we include some details about one of these simulations. We simulated a srs of size $n = 30$ from an exponential distribution X with mean $\mu_0 = 1.233151731$ (this value has been chosen to get $h'_{t_1,t_2}(\mu_0) = 0$ in Example 18 case (a)). The histogram of sample values is given in Fig. 1, we observe that data fit to an exponential distribution. The sample mean is $\bar{x}_n = 1.226735406$. Different parametric functions to estimate are considered.

Example 18. Let $h_{t_1,t_2}(\mu) = P_\mu[t_1 < X < t_2] = e^{-t_1/\mu} - e^{-t_2/\mu}$, with $0 < t_1 < t_2$. For $S_n \geqslant t_2$ the UMVUE of $h_{t_1,t_2}(\mu)$ is $T_n = (1 - \frac{t_1}{S_n})^{n-1} - (1 - \frac{t_2}{S_n})^{n-1}$. Consider the following cases:

(a) Let $(t_1, t_2) = (1, 1.5)$, $(h_{t_1,t_2}(\mu_0) = 0.14815,\ T_n = 0.15066)$. In this situation $h'_{t_1,t_2}(\mu_0) = 0$ and the case-deletion influence diagnostic shows a "quadratic" behaviour ($R^2 = 0.994$). See the scatter-plot $x - \widehat{b}_n(h(\mu_0), x)$ given in Fig. 1(a). Moreover the vertex is near the central values in the sample. The most important fact we observe in this case is that $\widehat{b}_n(h(\mu_0), x)$ is not an increasing function of $|x - \bar{x}_n|$. This situation is completely different from results we have for a linear function (see Example 16). Therefore it can be said that this case-deletion influence diagnostic can identify different influential observations depending on parametric function under consideration.

(b) Let $(t_1, t_2) = (1, 1.6)$, $(h_{t_1,t_2}(\mu_0) = 0.17133,\ T_n = 0.17428)$. In this case $h'_{t_1,t_2}(\mu_0) = 0.0048$ is a value close to zero. We observe a "quadratic" behaviour in the scatter-plot $x - \widehat{b}_n(h(\mu_0), x)$ given in Fig. 1(b) but no so clearly as the one in Fig. 1(a).

(c) Let $(t_1, t_2) = (1, 2.5)$, $(h_{t_1,t_2}(\mu_0) = 0.32573,\ T_n = 0.31981)$. In this case $h'_{t_1,t_2}(\mu_0) = 0.076$ is nonzero. We observe a nearly linear relationship in the scatter-plot $x - \widehat{b}_n(h(\mu_0), x)$ given in Fig. 1(c). Obviously $\widehat{b}_n(h(\mu_0), x)$ is not a linear function in x but its behaviour is linear for our sample values.

Example 19. Let $h_a(\mu) = (\mu - a)^3$, $a \in \mathbb{R}$. Then $T_n = \frac{1}{\beta_{3,n}} p_{3,n}(S_n, a)$. Taking $a = \mu_0$ we have $h'_a(\mu_0) = h''_a(\mu_0) = 0$. In spite of being a polynomial of degree 3, the scatterplot $x - \widehat{b}_n(h(\mu_0), x)$ in Fig. 1(d) shows a nearly linear behaviour.

5. Conclusions

In this paper two measures related to IF and influence analysis techniques are proposed to assess the possible effect of an observation on the UMVU estimate in the NEF-QVF. These are: the conditional bias of UMVUE to an observation and AMSC. The proposed measures take the advantage of having a clear meaning and being easy to introduce at an intermediate level. In fact the AMSC can be considered as an alternative to IF. A study in detail has been carried out: their properties, relationships and simulations have been discussed in deep. From this study we can conclude that these measures depend on kind of parametric function and the true and unknown value of the parameter.

As for the technique we use in this paper: expansions in terms of orthogonal polynomials for the UMVUE. It can be said that a new use of this technique has been proposed to get results on influence analysis for the UMVUE in certain classic parametric models (one-parameter NEFs with QVF).

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References

[1] F. Hampel, E.M. Ronchetti, P.J. Rousseuw, W.A. Stahel, Robust Statistics: The Approach Based on Influence Functions, Wiley, New York, 1986.
[2] M.G. Kim, Local influence in Box–Cox transformation for multivariate regression data, Comm. Statist. Theory Methods 1 (1996) 145–151.
[3] W.Y. Poon, M.L. Tang, Influence measure in maximum likelihood estimate for models of lifetime data, J. Appl. Stat. 6 (2001) 737–742.
[4] S.O. Nyangoma, W.K. Fung, R.C. Jansen, Identifying influential multinomial observations by perturbation, Comput. Statist. Data Anal. 50 (10) (2006) 2799–2821.

[5] J.L. Abbey, H.T. David, The construction of uniformly minimum variance unbiased estimators for exponential distributions, Ann. Math. Statist. 41 (4) (1970) 1217–1226.
[6] C.N. Morris, Natural exponential families with quadratic variance functions, Ann. Statist. 10 (1) (1982) 65–80.
[7] C.N. Morris, Natural exponential families with quadratic variance functions: Statistical theory, Ann. Statist. 11 (2) (1983) 515–529.
[8] F. López-Blázquez, A. Castaño-Martínez, Asymptotic properties of unbiased estimators in the natural exponential family with quadratic variance function, Sankhyā Ser. A 61 (2) (1999) 292–297.
[9] F. López-Blázquez, B. Salamanca-Miño, A reverse martingale property that characterizes the natural exponential family with quadratic variance function, Statist. Probab. Lett. 49 (1) (2000) 63–68.
[10] V.G. Voinov, M.S. Nikulin, Unbiased Estimators and Their Applications, vol. 1. Univariate Case, Kluwer Academic, 1993.
[11] I. Barranco-Chamorro, F. López-Blázquez, J.L. Moreno-Rebollo, Lower bounds for the variance in certain nonregular distributions, Statist. Probab. Lett. 50 (1) (2000) 75–81.
[12] D. Pommeret, Posterior variance for quadratic natural exponential families, Statist. Probab. Lett. 53 (4) (2001) 357–362.
[13] J. Muñoz-Pichardo, J. Muñoz-García, J.L. Moreno-Rebollo, R. Pino-Mejías, A new approach to influence analysis in linear models, Sankhyā Ser. A 57 (3) (1995) 393–409.
[14] J. Muñoz-Pichardo, J. Muñoz-Garcia, J.M. Fernández-Ponce, M.D. Jiménez-Gamero, Influence analysis in multivariate linear general models, Comm. Statist. Theory Methods 29 (3) (2000) 529–547.
[15] M.D. Jiménez-Gamero, J.L. Moreno-Rebollo, J.M. Muñoz-Pichardo, A.M. Muñoz-Reyes, Influence diagnostic in regression with complex designs through conditional bias, Test 14 (2) (2005) 515–542.
[16] J.L. Moreno-Rebollo, A.M. Muñoz-Reyes, J.M. Muñoz-Pichardo, Influence diagnostic in survey sampling: Conditional bias, Biometrika 84 (4) (1999) 923–928.
[17] R. Schlittgen, R. Schwabe, An alternative definition of the influence function, Statist. Probab. Lett. 51 (2001) 143–153.
[18] T. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[19] S. Portnoy, Asymptotic efficiency of minimum variance estimators, Ann. Statist. 5 (3) (1977) 522–529.